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Abstract
We consider the Fermi quantization of the classical damped harmonic oscillator
(dho). In past work on the subject, authors double the phase space of the dho in
order to close the system at each moment in time. For an infinite-dimensional
phase space, this method requires one to construct a representation of the CAR
algebra for each time. We show that the unitary dilation of the contraction
semigroup governing the dynamics of the system is a logical extension of the
doubling procedure, and it allows one to avoid the mathematical difficulties
encountered with the previous method.

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1. Introduction

The damped harmonic oscillator (dho) is a simple classical dissipative system which, upon
quantization, yields a useful example of an open quantum system [1]. Unstable particles can
be treated as an open system; these have been incorporated into quantum field theory in [2].
Additionally, modelling a quantum measurement as an open system permits one to regard
the reduction of a state as a continuous process [3]. A great deal of the extensive literature
regarding open systems and quantum stochastic processes is expressed in terms of density
operators; however, we shall avoid this formalism. Both Fermi and Bose quantizations of the
dho have been considered in the literature [4–7]. A technique central to these considerations is
the doubling of the degrees of freedom of the classical phase space, allowing one to effectively
close the system for each moment in time. The extra degrees of freedom function as a sink with
which the oscillator interacts and have been related to quantum noise effects [5]. The doubled
system as a whole has been expressed in the framework of quantum deformed algebras and
related to finite-temperature quantum systems [6, 7].

We shall restrict our discussion to the Fermi quantized dho, although the techniques
are similar in spirit for bosons. After introducing the classical dho, we will discuss the
doubling procedure in terms of the unitary dilation of a contraction operator at one instant
in time. We shall relate this technique to the representation of quasifree states over the
canonical anticommutation relation (CAR) algebra. Following the model developed in [3], we
introduce the unitary dilation of the contraction semigroup which describes the time evolution

of the dho. We then discuss the quantized dho in this framework in order to demonstrate the usefulness of this formulation. Our unified treatment of these structures will indicate the differences between the doubled dho and quasifree states, thereby demonstrating the success of the alternate contraction dilation.

2. Classical dho

A classical system can be described by a phase space M of even (or infinite) dimension whose elements specify momenta and position. We denote the complex structure on this space by J and the projection onto the momentum subspace as P. We intend the complex structure to generate harmonic oscillations necessitating the additional prescription

$$JP = (1 - P)J. \tag{1}$$

For the dho, we take the damping term to be linear in the momentum. Damped oscillations are then generated by the operator

$$Z = \omega J - 2\gamma P, \tag{2}$$

where ω is the natural (real) frequency of the oscillator and $\gamma > 0$ is the damping strength. The dynamics of a particular point in phase space $m \in M$ are governed by the semigroup

$$m(t) = T_t m, \qquad T_t = \exp[Zt], \tag{3}$$

for $t \geqslant 0$. Using the relation in (1), we may show that the generator satisfies the quadratic

$$(Z + \gamma 1)^2 = (\gamma^2 - \omega^2)1. \tag{4}$$

From this identity, we may write the exponential in (3) in a more tractable form which explicitly demonstrates the usual behaviour of a dho, depending upon the relative values of the damping factor and the natural frequency. Defining $\alpha = (\gamma^2 - \omega^2)^{1/2}$, we find

$$\exp[Zt] = \mathrm{e}^{-\gamma t} \left[\cosh \alpha t \, 1 + \frac{\sinh \alpha t}{\alpha}(Z + \gamma 1) \right]. \tag{5}$$

Should $\gamma^2 = \omega^2$, the case of critical damping, then we intend the above relation (5) to be taken in the limit of vanishing α; as such, we have

$$\lim_{\alpha \to 0} \frac{\sinh \alpha t}{\alpha} = t. \tag{6}$$

An underdamped oscillator is characterized by $\gamma^2 < \omega^2$ so that $\alpha = \mathrm{i}\omega_d$ is a purely imaginary number. In this case, we may write the hyperbolic functions in a more transparent form

$$\cosh \alpha t = \cos \omega_d t, \qquad \frac{\sinh \alpha t}{\alpha} = \frac{\sin \omega_d t}{\omega_d}; \tag{7}$$

hence, one has oscillations of frequency ω_d modulated by the decaying exponential. For an overdamped oscillator, $\gamma^2 > \omega^2$, one has real α so that the dynamics consist of decaying exponentials with two different decay constants.

3. Contraction dilation

Given equation (5), it is apparent that T_t is a contraction for all relevant t and, in fact, strongly converges to zero as t tends to infinity. From [8], we extract the following result concerning any contraction T on M. First, we define the isometric injection j of M into two copies of the phase space $\widetilde{M} = M \oplus M$

$$jm = m \oplus 0, \qquad m \in M. \tag{8}$$

Then, on the doubled phase space, we may construct from the contraction the orthogonal operator

$$U = \begin{pmatrix} T & (1 - TT^*)^{1/2} \\ (1 - T^*T)^{1/2} & -T^* \end{pmatrix}, \tag{9}$$

which satisfies

$$j^* U j = T. \tag{10}$$

In words, doubling the dimension of the phase space allows one to work with an orthogonal operator, instead of a contraction. The additional copy of the phase space can be regarded as a sink, or coupled oscillator.

As we are interested in the Fermi quantization of the dho, we work with the algebra $CAR(\widetilde{M})$ where the phase space is considered complex with complex structure $\widetilde{J} = J \oplus -J$. The creation operators $c(\tilde{m})$, linear in their argument, satisfy along with their adjoints the CAR

$$[c(\tilde{m})^*, c(\tilde{n})]_+ = \langle \tilde{m}, \tilde{n} \rangle, \qquad [c(\tilde{m}), c(\tilde{n})]_+ = 0, \tag{11}$$

where $[\cdot, \cdot]_+$ is the anticommutator. By definition, the operator (9) generates a Bogoliubov transformation of the algebra [9]. As this is true for any contraction on M, in particular T_t, we conclude there is a Bogoliubov transformation associated with the dilation U_t relating the Fock representation of $CAR(\widetilde{M})$ (at $t = 0$) and the dho on the doubled phase space at any fixed time t. This establishes a connection with the previous work on the subject [4–7].

In general, the operator U_t does not commute with the complex structure on the doubled space. We decompose the operator into the sum of a complex linear a_{U_t} and a conjugate linear b_{U_t} operator

$$a_{U_t} = \tfrac{1}{2}(U_t - \widetilde{J} U_t \widetilde{J}), \qquad b_{U_t} = \tfrac{1}{2}(U_t + \widetilde{J} U_t \widetilde{J}). \tag{12}$$

In the usual manner, we define the transformed creation operator

$$c_t(\tilde{m}) = c(a_{U_t} \tilde{m}) + c(b_{U_t} \tilde{m})^*; \tag{13}$$

these satisfy the CAR. From [10], we find that the Bogoliubov transformation is implementable if and only if b_{U_t} is Hilbert–Schmidt. Given an implementable transformation, one may construct from elements within the Fock representation a vacuum vector that is annihilated by $c_t(\tilde{m})^*$ [9]; in this sense, the vacuum vector can be thought of as a dynamic object. For a finite-dimensional phase space, this situation is assured. One may show that for fixed $t > 0$, the square of the Hilbert–Schmidt norm of b_{U_t} scales with the dimension of $\widetilde{M}$. As such, U_t is not implementable for an infinite-dimensional phase space. As a consequence, one must construct a different representation for each time t with no implementable means to change between any two representations. We remedy this situation below.

4. Quasifree states

First, we use similar language to elucidate the connection between this doubling procedure and general quasifree states over $CAR(M)$. A quasifree state φ is determined by the two-point correlation functions. The state can be characterized by two bounded operators R and (conjugate linear) S. These satisfy $0 \leqslant R = R^* \leqslant 1$ and $S^* = -S$ with the two-point correlation functions given by

$$\varphi[c(m)^* c(n)] = \langle m, Rn \rangle, \qquad \varphi[c(m)c(n)] = \langle Sm, n \rangle, \tag{14}$$

for $m, n \in M$. The representation theory of quasifree states over the CAR algebra is well developed [11, 12]; briefly, any quasifree state can be represented as a Fock state of the CAR

algebra over two copies of M. For the most general quasifree state, the connection between its representation and the doubling procedure used with the dho is most easily accessed through Araki's self-dual representation of the algebra [12]. We shall not discuss the details of this method because the required definitions and notation would take us too far afield. Instead, we restrict the discussion to quasifree states which are invariant under global $U(1)$ phase changes; these have $S = 0$, so that the state is characterized by R alone. We are justified in restricting our scope as these are the relevant states for physics.

Given that, we note from above that R is required to be a positive contraction. As such, its square root is defined, and $\sqrt{R}$ is also a contraction. We use the injection j defined in (8), set the complex structure $\widetilde{J} = i \oplus -i$, and unitarily dilate as before

$$V = \begin{pmatrix} \sqrt{R} & \sqrt{1-R} \\ \sqrt{1-R} & -\sqrt{R} \end{pmatrix}. \tag{15}$$

With this complex structure, one has

$$a_V = \begin{pmatrix} \sqrt{R} & 0 \\ 0 & -\sqrt{R} \end{pmatrix}, \qquad b_V = \begin{pmatrix} 0 & \sqrt{1-R} \\ \sqrt{1-R} & 0 \end{pmatrix}. \tag{16}$$

Using transformed creation operators, as in (13), acting on the Fock vacuum Ω, we calculate the two-point correlation function for elements of M injected into the doubled space

$$\langle c_V(jm)\Omega, c_V(jn)\Omega \rangle = \langle c(a_V jm)\Omega, c(a_V jn)\Omega \rangle = \langle m, Rn \rangle = \varphi[c(m)^* c(n)], \tag{17}$$

as one would expect.

The parallel with the quantized dho is particularly germane for KMS, or thermal, states. A KMS state φ_β, at inverse (positive) temperature β, is a quasifree state which describes a quantum system with Hamiltonian H in thermal equilibrium [13]. The defining property of this state is the commutation relation

$$\varphi_\beta[AB] = \varphi_\beta[BA_{i\beta}] \tag{18}$$

for operators A, B, where we have used a subscript to denote time evolution in the Heisenberg picture

$$A_t = U_t^* A U_t \tag{19}$$

with $U_t = e^{-iHt}$. Using the KMS condition (18) and the CAR (11), one can show that the state satisfies the two-point correlation functions in (14) with $R = (1 + e^{-\beta H})^{-1}$ and $S = 0$. Hence, we may represent the thermal state as a Fock state on two copies of the space for each temperature $\beta > 0$, an index reminiscent of time in the dho. This connection between the dho and thermal states was discussed in [7].

The Fermi quantized dho and quasifree states share a similar mathematical structure with regard to their representation on the doubled space. In particular, the analogy between the two is especially compelling for thermal states given that they are both indexed by positive numbers (temperature and time). However, the KMS states exhibit the much richer structure of Tomita–Takesaki theory [14]. As such, they necessarily have a unitary dynamical component in their definition. When quantizing the dho in the above manner, the dynamics of the oscillator are, in some sense, frozen out; that is, one has a different representation at each moment in time. Heuristically, the motivation of the doubling procedure is quite different for these two structures, making the physical connection between the two tenuous.

5. Semigroup dilation

Above we explicated how one may treat a single contraction as an orthogonal operator. If we are willing to inject the original phase space into a space even larger than $\widetilde{M}$, then it is possible to unitarily dilate the contraction semigroup T_t for all $t \geqslant 0$. In what follows, we shall maintain the same notation as above in order to make clear that this is a logical extension of the doubling procedure. The technique that follows can be found in [8]; the application of this theory to the dho was expounded upon in [3]. An explicit application of dilation theory to the finite-dimensional bosonic dho is available in [15]. The space into which we inject M is $\widehat{M} = L^2(\mathbb{R}, PM)$. The injection $j : M \to \widehat{M}$ is given by

$$(jm)(t) = 2\sqrt{\gamma}\,\Theta(t)\,PT_tm, \tag{20}$$

with $\Theta(t)$ the Heaviside function. This map can be shown to be an isometry. Time translation of the elements of the space is given by the unitary operator

$$(U_t\widehat{m})(s) = \widehat{m}(s + t). \tag{21}$$

Analogous to equation (10), the following holds by construction

$$j^*U_tj = T_t. \tag{22}$$

Using this technique, one no longer needs to bother with Bogoliubov transformations in the above prescription or their implementation. Though $\widehat{M}$ is a much larger space than the doubled one, the dynamics of the dho plus environment are now unitary. In order to keep track of the dho in this space, we employ the projection $Q = jj^*$ on $\widehat{M}$. One can then make the orthogonal decomposition

$$\widehat{M} = jM \oplus (1 - Q)\widehat{M}, \tag{23}$$

which delineates the oscillator and the environment. This method of dilation also exhibits a richer structure than that of the doubling procedure. For instance, we may use the Fourier transform to ascertain the energy spread of an element of jM. As energy is dual to time, we have

$$(\mathcal{F}jm)(E) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iEt}(jm)(t)\, dt. \tag{24}$$

After some manipulation [3], this can be shown to be

$$(\mathcal{F}jm)(E) = \sqrt{\frac{2\gamma}{\pi}}\,\frac{1}{E^2 - \omega^2 - i2E\gamma}\,P(\omega J + iE)m, \tag{25}$$

which has an amplitude reminiscent of the relativistic Breit–Wigner amplitude for unstable particles (cf [16]).

One may quantize this total closed system in the usual manner by considering $\mathrm{CAR}(\widehat{M})$. The Fock representation is now adequate to describe the quantum dho. The vacuum vector Ω is a stationary state, and the dynamics are unitary. We may still address the dissipative nature of the dho in this space. For this, we consider the second quantization $\widetilde{Q}$ of the projection Q which obeys the commutation relation

$$[\widetilde{Q}, c(\widehat{m})] = c(Q\widehat{m}), \tag{26}$$

for $\widehat{m} \in \widehat{M}$. Recalling the time dependence in the Heisenberg picture (19), we calculate the following expectation value in the Fock representation:

$$\langle c(jm)\Omega, \widetilde{Q}(t)c(jn)\Omega\rangle = \langle jm, Q(t)jn\rangle = \langle j^*U_tjm, j^*U_tjn\rangle = \langle m, T_t^*T_tn\rangle, \tag{27}$$

for $n, m \in M$. This exhibits the behaviour that one would expect. The decomposition of the extended phase space $\widehat{M}$ in (23) suggests that, upon quantization, the Fock space of the

environment can easily be extricated from that of the oscillator. However, given the presence of the Heaviside function in the definition of the injection j in (20), it is clear that the associated projection $\widetilde{Q}$ is not a trivial operator, making the particle-plus-bath description difficult.

For the case of the critically damped oscillator, we note that given a proper choice of initial conditions one can model a collection of unstable particles as in [2]. For elements of the (nontrivial) subspace $N = \ker(Z + \gamma 1)$, the evolution of the critically damped oscillator is strictly exponential, $T_t|_N = e^{-\gamma t}$. As a result, for $n \in N$, the time dependence of the projection onto the oscillator subspace exhibits exponential decay

$$\langle c(jn)\Omega, \widetilde{Q}(t)c(jn)\Omega \rangle = e^{-2\gamma t}. \tag{28}$$

This is another method by which unstable particles can be included in quantum field theory.

6. Conclusion

In summation, we feel that the method of unitary dilation of the contraction semigroup is a more effective means with which to consider the quantized dho. Rather than closing the open system at each instant in time by doubling the dimension of the the phase space, we close the entire system at once for all future time. The advantages of this method are unitary dynamics and the need for only one Fock representation with a stationary vacuum vector. The drawback for this approach is that the analogy between the dho and thermal states is no longer valid; however, we feel that our exposition demonstrates that the two systems are conceptually and physically disparate, lessening the significance of this shortcoming.

Acknowledgments

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